AD-A102 989 MISSISSIPPI STATE UNIV MISSISSIPPI STATE F/G 20/4

THE GENERATION OF THREE-DIMENSIONAL BODY-FITTED COORDINATE SYST--ETC(U)

JUL 81 Z U WARSI, C W MASTIN, J F THOMPSON AFOSR-80-0185

UNCLASSIFIED AFOSR-TR-81-0635 NL

1 OF 1
AD
A102989

END
DATE
FILMED
9-81
DTIC

The Generation of Three-Dimensional Body-Fitted Coordinate Systems For Viscous Flow Problems

By
Z.U.A. Warsi*, C.W. Mastin** and J.F. Thompson***
Mississippi State University, Mississippi State, MS 39762

Abstract

A set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies have been proposed[1]. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when the equations depending on all the three independent variables are considered.

The resulting equations have been programmed for the numerical solution of the equations on the Cray-computer. Much of the time has been spent on the generation of surface coordinates for an aircraft fuselage by spline and various other methods. Once these surface coordinates have been accurately established, the proposed field equations will be solved for the region between the fuselage or other body and another arbitrarily selected outer surface (e.g., a sphere). The spline method of the body-coordinates is discussed below.

It has been established that spline interpolation can be used to construct a computational grid about a simple wing-fuselage configuration. For generality, the components of the configuration are

---

*Department of Aerospace Engineering (principal investigator)

**Department of Mathematics (co-principal investigator)

***Department of Aerospace Engineering.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER (18) AFOSR-TR-81-0635	2. GOVT ACCESSION NO. AD-A102989	3. RECIPIENT'S CATALOG NUMBER (9)
4. TITLE (and Subtitle) (6) The Generation of Three-Dimensional Body-Fitted Coordinate Systems for Viscous Flow Problems.		5. TYPE OF REPORT & PERIOD COVERED Interim May 1980 - Apr[redacted]981
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Z.U.A. Warsi, C.W. Mastin and J.F. Thompson (15)		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0185
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mississippi State University Department of Aerospace Engineering Mississippi State, MS 39762		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (17) 2304/A3 and 2307/A1 61102F A3 A1
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research /NM Building 410, Bolling AFB, D.C. 20332 (11)		12. REPORT DATE Jul[redacted]981
		13. NUMBER OF PAGES 46
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 47 (16) 2304, 2307		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Grid Generation, Curvilinear Coordinates, Numberical Methods, Computational Fluid Dynamics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates is reported with the generic equations as the Beltrami equations in the surface coordinates. Numerical computation with the aircraft fuselage as the inner body is in progress, with the proposed equations as the generating equations.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

410 ...

defined by algebraic equations. Thus the major dimensions of the body can be modified by changing only a few input parameters. The progress in this area is summarized in the attached manuscript which has been accepted for presentation at Mathematical Modeling in Los Angeles, July 29-31.

There are many surface description routines available which are used in the design of aircraft and other complex bodies. A more recent effort is to take one of these routines and build a computational grid about the body using the given surface description. The work in this area has been mainly in reprograming and little tangible results can be reported at the present time.

Papers Written or Presented

1. "A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes," Z.U.A. Warsi. MSSU-EIRS-80-7, Oct. (1980).

2. "Tensors and Differential Geometry Applied to Analytic and Numerical Coordinate Generation," Z.U.A. Warsi. MSSU-EIRS-81-1, Jan. (1981).

3. "Grid Generation for Aircraft Configuration," C.W. Mastin and J.F. Thompson.

A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes*

by

Z. U. A. Warsi†
Department of Aerospace Engineering
Mississippi State University
Mississippi State, MS 39762

Abstract

Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies is presented. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when equations depending on all three independent variables are considered. The method also allows, in a straight forward manner, the possibility of coordinate contraction in the desired regions.

An exact solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere with coordinate contraction is presented to demonstrate that by using these equations it is possible to generate three-dimensional coordinates between analytically specified surfaces of simple forms by analytical means.

The fundamental constraining equations which have been adopted for the generation of coordinates are $\Delta_2\xi = 0$ and $\Delta_2\eta = 0$, where Δ_2 is the surface Beltrami operator of the second order.

*Research supported in part by the Air Force Office of Scientific Research, under Grant AFOSR No. 80-0185.

†Professor

1. Introduction

At present a number of techniques are under active development for the generation of three-dimensional body-oriented coordinate systems for use in the numerical solution of the Navier-Stokes equations and other field equations where the exact specification of the boundary conditions is of prime importance. Among these efforts two easily discernable groups can be formed, (i) algebraic methods, and (ii) the elliptic equations method. In the first group the grid points in space are obtained by some interpolation or blending functions scheme which depends on the given boundary data. The choice of the interpolation scheme or of the blending functions is crucial in achieving a desired order of smoothness and distribution of the grid points in space. This line of effort has actively been considered by Eiseman [1,2], Smith and Weigel [3], and Eriksson [4]. In the second group of efforts, a set of three poisson equations in the curvilinear coordinates are first inverted and then solved for the Cartesian coordinates under the prescribed values at the given boundaries. Thus in essence all the methods of the second group are a straight forward extension of the work of Thompson et al [5] in two dimensions. Research in this area has been conducted by Mastin et al [6], Yu [7], Ghia et al [8], and Graves [9].

At this stage of research it is premature to compare the two groups since neither of them have been fully investigated for their inherent potentials. However, based on the success of the differential equations approach in two dimensions, e.g. [5], it is desirable to further investigate the elliptic equations approach for the generation of coordinates.

The elliptic equations approach presented in this paper is different from the approaches adopted in the previously cited works, i.e., References

[6] - [9]. The proposed method depends heavily on the formulae of Gauss and on the concept of principal curvatures of a surface. It has been shown that a fruitful arrangement of the classical differential-geometric results can yield a method which is easily programmable on a computing machine, and which at any time solves a two-dimensional partial differential equation of the form used in Ref. [5]. In this paper only the theoretical development of the method along with a technique to redistribute the coordinate surfaces near the inner boundary surface has been considered. The developed equations have been solved for the generation of three-dimensional coordinates between an inner prolate ellipsoid and an outer spherical surface in an exact analytic form.

2. Notation and Collection of Formulas

In what follows, the general coordinates are denoted as x^i (i = 1, 2,3). However when an expression has been expanded out in full and there is no use for an index notation, we have set

$$x^1 = \xi \ , \ x^2 = \eta \ , \ x^3 = \zeta \ .$$

The derivatives of the position vector $\underset{\sim}{r} = (x,y,z)$ are denoted as

$$\underset{\sim}{r}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i} \ , \ \underset{\sim}{r}_{ij} = \frac{\partial^2 \underset{\sim}{r}}{\partial x^i \partial x^j} \ .$$

The covariant components of the metric tensor are

$$g_{ij} = \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j \tag{2.1}$$

while the contravariant components are given by

$$g^{ij} g_{kj} = \delta^i_k \tag{2.2}$$

Thus in three dimensions

$$g = \det(g_{ij})$$

$$= g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} \tag{2.3}$$

Writing

$$G_1 = g_{22}g_{33} - (g_{23})^2$$

$$G_2 = g_{11}g_{33} - (g_{13})^2$$

$$G_3 = g_{11}g_{22} - (g_{12})^2$$

$$G_4 = g_{13}g_{23} - g_{12}g_{33}$$

$$G_5 = g_{12}g_{23} - g_{13}g_{22}$$

$$G_6 = g_{12}g_{13} - g_{11}g_{23} \tag{2.4}$$

we have

$$\begin{aligned} g^{11} &= G_1/g \ , \ g^{22} = G_2/g \ , \ g^{33} = G_3/g \\ g^{12} &= G_4/g \ , \ g^{13} = G_5/g \ , \ g^{23} = G_6/g \end{aligned} \tag{2.5}$$

The Christoffel symbols based on the metric g_{ij} are

$$\Gamma^i_{jk} = g^{i\ell}[jk,\ell]$$

where

$$\begin{aligned} [jk,\ell] &= \underset{\sim}{r}_{jk} \cdot \underset{\sim}{r}_{\ell} \\ &= \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right) \end{aligned} \tag{2.6}$$

and repeated lower and upper indices imply summation. In the sequel we have also used the surface Christoffel symbols which have been denoted as $T^{\alpha}_{\beta\gamma}$, where the Greek indices range over (1,2) or (3,1) or (2,3).

The coordinate which is held fixed to account for the surface geometry is denoted by a superscript in parentheses. Thus the unit normal vector on the surface ν = const. is given by

$$n^{(\nu)} = (r_\alpha \times r_\beta)/|r_\alpha \times r_\beta| \tag{2.7}$$

where

$$\left.\begin{array}{ll} \nu = 1 : \alpha = 2 \ , \ \beta = 3 & (\text{surface } x^1 = \text{const.}) \\ \nu = 2 : \alpha = 3 \ , \ \beta = 1 & (\text{surface } x^2 = \text{const.}) \\ \nu = 3 : \alpha = 1 \ , \ \beta = 2 & (\text{surface } x^3 = \text{const.}) \end{array}\right\} \tag{2.8}$$

The rectangular Cartesian components of $n^{(\nu)}$ are denoted as

$$n^{(\nu)} = (X^{(\nu)} \ , \ Y^{(\nu)} \ , \ Z^{(\nu)}) \tag{2.9}$$

The coefficients of the second fundamental form are denoted by $S^{(\nu)}$, $T^{(\nu)}$ and $U^{(\nu)}$ defined as

$$\left.\begin{array}{l} S^{(\nu)} = n^{(\nu)} \cdot r_{\alpha\alpha} \quad (\text{no sum on } \alpha) \\ T^{(\nu)} = n^{(\nu)} \cdot r_{\alpha\beta} \\ U^{(\nu)} = n^{(\nu)} \cdot r_{\beta\beta} \quad (\text{no sum on } \beta) \end{array}\right\} \tag{2.10}$$

where (ν,α,β) are in the cyclic permutations of (1,2,3), in this order.

The partial derivatives of the second order are expressible in terms of the first order as

$$r_{ij} = \Gamma^k_{ij} r_k \tag{2.11}$$

For a surface on which one of the coordinates is fixed, the Gauss' equations are

$$\left.\begin{aligned} \underset{\sim}{r}_{\alpha\alpha} &= T^{\gamma}_{\alpha\alpha}\underset{\sim}{r}_{\gamma} + S^{(\nu)}\underset{\sim}{n}^{(\nu)} \\ \underset{\sim}{r}_{\alpha\beta} &= T^{\gamma}_{\alpha\beta}\underset{\sim}{r}_{\gamma} + T^{(\nu)}\underset{\sim}{n}^{(\nu)} \\ \underset{\sim}{r}_{\beta\beta} &= T^{\gamma}_{\beta\beta}\underset{\sim}{r}_{\gamma} + U^{(\nu)}\underset{\sim}{n}^{(\nu)} \end{aligned}\right\} \tag{2.12}$$

Where (ν,α,β) are in the permutational sequences of $(1,2,3)$ as shown in (2.8), and the repeated index γ implies summation on the two indices of a surface.

The sum of the principal curvatures of the surface ν = const. is, [10],

$$k_1^{(\nu)} + k_2^{(\nu)} = (g_{\alpha\alpha}U^{(\nu)} - 2g_{\alpha\beta}T^{(\nu)} + g_{\beta\beta}S^{(\nu)})/G_\nu \tag{2.13}$$

where in writing equation (2.13) for a particular value of ν, use must be made of Eqs. (2.8) and (2.10). We now introduce two second order surface differential operators by using (2.8), which for ν = const. are

$$D^{(\nu)} \equiv g_{\beta\beta}\partial_{\alpha\alpha} - 2g_{\alpha\beta}\partial_{\alpha\beta} + g_{\alpha\alpha}\partial_{\beta\beta} \tag{2.14}$$

$$\begin{aligned} \Delta_2^{(\nu)} \equiv \frac{1}{\sqrt{G_\nu}}[\partial_\alpha\{\frac{1}{\sqrt{G_\nu}}(g_{\beta\beta}\partial_\alpha - g_{\alpha\beta}\partial_\beta)\} \\ + \partial_\beta\{\frac{1}{\sqrt{G_\nu}}(g_{\alpha\alpha}\partial_\beta - g_{\alpha\beta}\partial_\alpha)\}] \end{aligned} \tag{2.15}$$

As is well known, the operator Δ_2 is the Beltrami differential operator of the second order [10].

The three space Christoffel symbols which have been referred to in the next section are given below.

$$\Gamma_{11}^3 = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \xi} + G_6(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta})]$$

$$+ \frac{G_3}{g}(x_{\xi\xi}x_\zeta + y_{\xi\xi}y_\zeta + z_{\xi\xi}z_\zeta) \tag{2.16}$$

$$\Gamma_{12}^3 = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \eta} + G_6 \frac{\partial g_{22}}{\partial \xi}]$$

$$+ \frac{G_3}{g}(x_{\xi\eta}x_\zeta + y_{\xi\eta}y_\zeta + z_{\xi\eta}z_\zeta) \tag{2.17}$$

$$\Gamma_{22}^3 = \frac{1}{2g}[G_5(2 \frac{\partial g_{11}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_6 \frac{\partial g_{22}}{\partial \eta}]$$

$$+ \frac{G_3}{g}(x_{\eta\eta}x_\zeta + y_{\eta\eta}y_\zeta + z_{\eta\eta}z_\zeta) \tag{2.18}$$

3. Formulation of the Problem

The principal idea of the method to be presented is to generate a series of surfaces on each of which a certain a'priori chosen variable or coordinate is kept fixed. Each surface to be generated starts from a given curve of the inner body and ends on the corresponding curve of the outer boundary, cf. Fig. 1. A routine, preferably a spline fit, can then be used to join the successive generated surfaces so as to have a smooth three-dimensional computational net for solving other physical field equations.

To illustrate the method, we take $\nu = 3$ or $x^3 = \zeta$ = constant on each surface to be generated. Thus $\alpha = 1$ and $\beta = 2$, viz., α and β respectively correspond to the coordinates $x^1 = \xi$ and $x^2 = \eta$. For the sake of brevity of notation we will not use the superscript (3) unless it becomes necessary. Thus from (2.10)

$$S = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} \; , \; T = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} \; , \; U = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} \tag{3.1}$$

where

$$\underset{\sim}{n} = \underset{\sim}{i}X + \underset{\sim}{j}Y + \underset{\sim}{k}Z \tag{3.2}$$

Equations (2.12) are

$$\underset{\sim}{r}_{\xi\xi} = T^{\gamma}_{11}\underset{\sim}{r}_{\gamma} + S\underset{\sim}{n} \tag{3.3}$$

$$\underset{\sim}{r}_{\xi\eta} = T^{\gamma}_{12}\underset{\sim}{r}_{\gamma} + T\underset{\sim}{n} \tag{3.4}$$

$$\underset{\sim}{r}_{\eta\eta} = T^{\gamma}_{22}\underset{\sim}{r}_{\gamma} + U\underset{\sim}{n} \tag{3.5}$$

From (2.14) and (2.15) the operators $D^{(3)}$ and $\Delta_2^{(3)}$ are

$$D \equiv g_{22} \frac{\partial^2}{\partial \xi^2} - 2g_{12} \frac{\partial^2}{\partial \xi \partial \eta} + g_{11} \frac{\partial^2}{\partial \eta^2} \tag{3.6}$$

$$\Delta_2 \equiv \frac{1}{\sqrt{G_3}}[\frac{\partial}{\partial \xi}\{\frac{1}{\sqrt{G_3}}(g_{22} \frac{\partial}{\partial \xi} - g_{12} \frac{\partial}{\partial \eta})\}$$

$$+ \frac{\partial}{\partial \eta}\{\frac{1}{\sqrt{G_3}}(g_{11} \frac{\partial}{\partial \eta} - g_{12} \frac{\partial}{\partial \xi})\}] \tag{3.7}$$

We now multiply equations (3.3) - (3.5) respectively by g_{22}, $-2g_{12}$, g_{11} adding and using equations (2.13) and (2.15) to have

$$D\underset{\sim}{r} + G_3(\underset{\sim}{r}_\xi \Delta_2 \xi + \underset{\sim}{r}_\eta \Delta_2 \eta) = G_3 \underset{\sim}{n}(k_1 + k_2) \tag{3.8}$$

where

$$\left.\begin{aligned} \Delta_2 \xi &= \frac{1}{G_3}(2g_{12}T^1_{12} - g_{22}T^1_{11} - g_{11}T^1_{22}) \\ \Delta_2 \eta &= \frac{1}{G_3}(2g_{12}T^2_{12} - g_{22}T^2_{11} - g_{11}T^2_{22}) \\ G_3 &= g_{11}g_{22} - (g_{12})^2 \end{aligned}\right\} \tag{3.9}$$

To obtain an expression for $k_1 + k_2$ consider equation (2.11) and utilize the property that $\underset{\sim}{n}$ is orthogonal to $\underset{\sim}{r}_\xi$ and $\underset{\sim}{r}_\eta$, so that

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} = \Gamma^3_{11}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} = \Gamma^3_{12}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} = \Gamma^3_{22}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta) \tag{3.10}$$

where all the derivatives with respect to ζ are evaluated at ζ = constant. Multiplying Eq. (3.8) scalarly by $\underset{\sim}{n}$ and using (3.10), we get

$$G_3(k_1 + k_2) = (\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{3.11}$$

<u>We now propose the following deterministic problem:</u> Let ξ and η be the surface coordinates on the surface ζ = constant, subject to the constraints

$$\begin{aligned} \Delta_2\xi &= 0 \\ \Delta_2\eta &= 0 \end{aligned} \tag{3.12}$$

Then the Cartesian coordinates x,y,z of the surface satisfy the differential equations

$$D\underset{\sim}{r} = G_3(k_1 + k_2)\underset{\sim}{n} \tag{3.13}$$

The three scalar differential equations for the generation of the Cartesian coordinates are then

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = XR \tag{3.14}$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = YR \tag{3.15}$$

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = ZR \tag{3.16}$$

where

$$R = (Xx_\zeta + Yy_\zeta + Zz_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{3.17}$$

and

$$\left.\begin{aligned} X &= (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3} \\ Y &= (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3} \\ Z &= (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3} \end{aligned}\right\} \tag{3.18}$$

Equations (3.14) - (3.16) form a quasilinear system of partial differential equations in which the components of $\underset{\sim}{r}_\zeta$ are assumed to be known. Since the values of x,y,z are known on the basic inner and outer boundaries (denoted at B and ∞ respectively in Fig. 1), a suitable way of prescribing $\underset{\sim}{r}_\zeta$ can be to take

$$\underset{\sim}{r}_\zeta = f_1(\eta)(\underset{\sim}{r}_\zeta)_B + f_2(\eta)(\underset{\sim}{r}_\zeta)_\infty \tag{3.19}$$

where $f_1(\eta)$ and $f_2(\eta)$ are suitable weights having the properties

$$f_1(\eta_B) = 1 \ , \ f_2(\eta_B) = 0$$

$$f_1(\eta_\infty) = 0 \ , \ f_2(\eta_\infty) = 1$$

For exposing the essential nonlinear terms in the factor R we refer to Eqs. (2.16) - (2.18) in which the $\underset{\sim}{r}_\zeta$ terms have been collected separately.

Referring to Figure 2, we now solve Eqs. (3.14) - (3.16) for each ζ = const., on a rectangular plane by prescribing the values of x, y and z on the lower side (C_1) and upper side (C_2) which represent the curves on B and ∞ respectively. The sides C_3 and C_4 are the cut lines on which periodic boundary conditions are to be imposed. The preceding analysis thus completes the formulation of the problem.

4. Coordinate Transformation (Contraction)

For the purpose of generating coordinates between the space of the inner and outer boundary which can be distributed in a desired manner, we consider a coordinate transformation from $\xi \to \chi$ and $\eta \to \sigma$. Let

$$\begin{aligned} \xi &= \xi(\chi) + \xi_o \\ \eta &= \eta(\sigma) + \eta_B \end{aligned} \qquad (4.1)$$

then

$$\begin{aligned} \xi &= \xi_o \text{ at } \chi = \chi_o \text{ , } \xi(\chi_o) = 0 \\ \eta &= \eta_B \text{ at } \sigma = \sigma_B \text{ , } \eta(\sigma_B) = 0 \end{aligned} \qquad (4.2)$$

Writing

$$\lambda(\chi) = \frac{d\xi}{d\chi} \text{ , } \theta(\sigma) = \frac{d\eta}{d\sigma}$$

and denoting the transformed metric tensor as $\bar{g}_{ij}$, we have

$$\left.\begin{aligned} g_{11} &= \bar{g}_{11}/\lambda^2 \text{ , } \bar{g}_{11} = x_\chi^2 + y_\chi^2 + z_\chi^2 \\ g_{12} &= \bar{g}_{12}/\theta\lambda \text{ , } \bar{g}_{12} = x_\chi x_\sigma + y_\chi y_\sigma + z_\chi z_\sigma \\ g_{22} &= \bar{g}_{22}/\theta^2 \text{ , } \bar{g}_{22} = x_\sigma^2 + y_\sigma^2 + z_\sigma^2 \\ G_3 &= \bar{G}_3/\theta^2\lambda^2 \text{ , } \bar{G}_3 = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2 \\ X &= \bar{X} \text{ , } Y = \bar{Y} \text{ , } Z = \bar{Z} \end{aligned}\right\} \qquad (4.3)$$

$$k_1 + k_2 = \bar{k}_1 + \bar{k}_2$$

$$R = \bar{R}/\theta^2\lambda^2$$

Further noting that

$$\left.\begin{aligned} \underset{\sim}{r}_{\xi\xi} &= \left(\underset{\sim}{r}_{\chi\chi} - \frac{\underset{\sim}{r}_{\chi}\lambda_{\chi}}{\lambda}\right)/\lambda^2 \\ \underset{\sim}{r}_{\xi\eta} &= \underset{\sim}{r}_{\chi\sigma}/\theta\lambda \\ \underset{\sim}{r}_{\eta\eta} &= \left(\underset{\sim}{r}_{\sigma\sigma} - \frac{\underset{\sim}{r}_{\sigma}\theta_{\sigma}}{\sigma}\right)/\theta^2 \end{aligned}\right\} \tag{4.4}$$

Using (4.3) and (4.4) in Eqs. (3.14) - (3.16), we have

$$\bar{g}_{22}x_{\chi\chi} - 2\bar{g}_{12}x_{\chi\sigma} + \bar{g}_{11}x_{\sigma\sigma} = Px_{\chi} + Qx_{\sigma} + \bar{X}\bar{R} \tag{4.5}$$

$$\bar{g}_{22}y_{\chi\chi} - 2\bar{g}_{12}y_{\chi\sigma} + \bar{g}_{11}y_{\sigma\sigma} = Py_{\chi} + Qy_{\sigma} + \bar{Y}\bar{R} \tag{4.6}$$

$$\bar{g}_{22}z_{\chi\chi} - 2\bar{g}_{12}z_{\chi\sigma} + \bar{g}_{11}z_{\sigma\sigma} = Pz_{\chi} + Qz_{\sigma} + \bar{Z}\bar{R} \tag{4.7}$$

where

$$\begin{aligned} P &= \frac{\bar{g}_{22}}{\lambda}\lambda_{\chi} \\ Q &= \frac{\bar{g}_{11}}{\theta}\theta_{\sigma} \end{aligned} \tag{4.8}$$

Thus, by choosing λ and θ arbitrarily we can redistribute the coordinates in the desired manner. An example of this choice is given in the next section.

5. An Analytical Example of Coordinate Generation

In this section we shall consider the problem of coordinate generation between a prolate ellipsoid and a sphere with coordinate contraction near the inner surface. This problem yields an exact solution of the equations (4.5) - (4.7).

Let $\eta = \eta_B$ and $\eta = \eta_\infty$ be the inner prolate ellipsoid and the outer sphere respectively. The coordinates which vary on these two surfaces are ξ and ζ. We now envisage a net of lines ξ = const. and ζ = const. on these two surfaces. A curve C_1 on the inner surface designated as $\zeta = \zeta_o$ is

$$\left.\begin{aligned} x &= \cosh\eta_B \cos\zeta_o \\ y &= \sinh\eta_B \sin\zeta_o \cos\xi \\ z &= \sinh\eta_B \sin\zeta_o \sin\xi \end{aligned}\right\} \tag{5.1}$$

Similarly, the curve C_2 corresponding to $\zeta = \zeta_o$ on the outer surface is

$$\left.\begin{aligned} x &= e^{\eta_\infty} \cos\zeta_o \\ y &= e^{\eta_\infty} \sin\zeta_o \cos\xi \\ z &= e^{\eta_\infty} \sin\zeta_o \sin\xi \end{aligned}\right\} \tag{5.2}$$

Based on the forms of the functions x,y,z in (5.1) and (5.2), we assume the following forms of x,y,z for the surface $\zeta = \zeta_o$:

$$\left.\begin{aligned} x &= f(\sigma)\cos\zeta_o \\ y &= \phi(\sigma)\sin\zeta_o\cos\xi \\ z &= \phi(\sigma)\sin\zeta_o\sin\xi \end{aligned}\right\} \qquad (5.3)$$

The boundary conditions for f and ϕ are

$$\left.\begin{aligned} f(\sigma_B) &= \cosh\eta_B \\ f(\sigma_\infty) &= e^{\eta_\infty} \\ \phi(\sigma_B) &= \sinh\eta_B \\ \phi(\sigma_\infty) &= e^{\eta_\infty} \end{aligned}\right\} \qquad (5.4)$$

Calculating the various derivatives, metric coefficients, and all other data needed in the equations (4.5) - (4.7), we get on substitution an equation which has $\sin^2\zeta_o$ and $\cos^2\zeta_o$. Equating to zero the coefficients of $\sin^2\zeta_o$ and $\cos^2\zeta_o$, we obtain

$$\frac{f''}{f'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.5)$$

$$\frac{\phi''}{\phi'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.6)$$

where a prime denotes differentiation with respect to σ. On direct integration of Eqs. (5.5) and (5.6) under the boundary conditions (5.4), we get

$$f(\sigma) = Ae^{B\eta(\sigma)} + C \tag{5.7}$$

$$\phi(\sigma) = De^{B\eta(\sigma)} \tag{5.8}$$

where

$$A = \frac{(e^{\eta_\infty} - \cosh\eta_B)\sinh\eta_B}{e^{\eta_\infty} - \sinh\eta_B} \tag{5.9a}$$

$$B = \ln\left[\frac{e^{\eta_\infty}}{\sinh\eta_B}\right]^{1/(\eta_\infty - \eta_B)} \tag{5.9b}$$

$$C = \frac{e^{\eta_\infty}(\cosh\eta_B - \sinh\eta_B)}{e^{\eta_\infty} - \sinh\eta_B} \tag{5.9c}$$

$$D = \sinh\eta_B \tag{5.9d}$$

As an application we may take [11]

$$\xi(\chi) = a\chi$$

$$\eta(\sigma) = b(\sigma - \sigma_B)K^\sigma$$

where a and b are constants. Since at η_∞,

$$\eta(\sigma_\infty) = \eta_\infty - \eta_B$$

hence

$$\eta(\sigma) = \frac{(\eta_\infty - \eta_B)(\sigma - \sigma_B)}{(\sigma_\infty - \sigma_B)}K^{(\sigma - \sigma_\infty)}$$

By taking a value of K slightly greater than one ($K = 1.05$ or 1.1), we can have sufficient contraction of coordinates near the inner surface.

For the chosen problem, since the dependence on ζ is simple, we find that the coordinates between a prolate ellipsoid and a sphere are

$$x = [Ae^{B\eta(\sigma)} + C]\cos\zeta$$

$$y = De^{B\eta(\sigma)}\sin\zeta\cos\xi$$

$$z = De^{B\eta(\sigma)}\sin\zeta\sin\xi$$

where A, B, C, and D are given in equation (5.9).

6. Conclusions

A new method for the generation of three-dimensional coordinates between two arbitrary shaped bodies has been presented. The method is based on some simple differential-geometric concepts such as the equations of Gauss and the expressions for the principal curvatures of a surface. The simplicity of the method lies in solving, at one time, only three partial differential equations of the two-dimensional type. This aspect is bound to reduce the working core requirements for a given problem on a computing machine. Finally the method allows, in a very direct fashion, the possibility of coordinate redistribution in the desired regions (cf. Eqs. (4.5) - (4.7)).

An analytic solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere has been presented. This example shows that one can generate coordinates between two analytically specified surfaces of simple forms by exact solutions of the proposed equations.

In this paper the fundamental equations which form a set of constraints for the generation of coordinates in the surface are

$$\Delta_2 \xi = 0$$

$$\Delta_2 \eta = 0 ,$$

where Δ_2 is the surface Beltrami operator of the second order. It must be noted that Δ_2 is neither a Laplace operator in the Cartesian plane (x,y), nor in the Cartesian space (x,y,z). However, in the case of a Cartesian plane (x,y), when there is no dependence on z, Δ_2 reduces to the Laplace operator ∇^2.

References

1. P. R. Eisemann, "A Multi-Surface Method of Coordinate Generation," Journal of Computational Physics, 33 (1979), 118.

2. P. R. Eisemann and R. E. Smith, "Mesh Generation Using Algebraic Techniques," Workshop on Numerical Generation Techniques for Partial Differential Equations, NASA Langley Research Center, Hampton, Virginia, Oct. 6-7, (1980).

3. R. E. Smith and B. L. Weigel, "Analytical and Approximate Boundary-Fitted Coordinate Systems," AIAA paper 80-0192 (1980).

4. L.-E. Eriksson, "Three-Dimensional Spline-Generated Coordinate Transformations for Grids Around Wing-Body Configurations," Workshop on Numerical Generation Techniques for Partial Differential Equations, NASA Langley Research Center, Hampton, Virginia, Oct. 6-7, (1980).

5. J. F. Thompson, F. C. Thames, and C. W. Mastin, "Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing any Number of Arbitrary Two-Dimensional Bodies," Journal of Computational Physics, 15 (1974), 299.

6. C. W. Mastin and J. F. Thompson, "Transformation of Three-Dimensional Regions onto Rectangular Regions by Elliptic Systems," Numerische Mathematik, 29 (1978), 397.

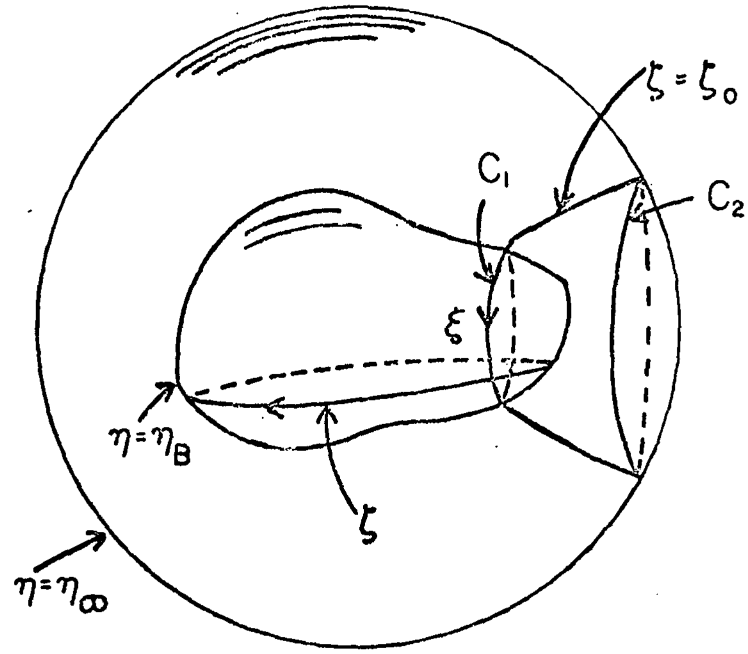
7. N. J. Yu, "Grid Generation and Transonic Flow Calculations for Three-Dimensional Configurations," AIAA paper 80-1391 (1980).

8. U. Ghia and K. N. Ghia, "Numerical Generation of a System of Curvilinear Coordinates for Turbine Cascade Flow Analysis," Rep. No. AFL 75-4-17, University of Cincinnati (1975).

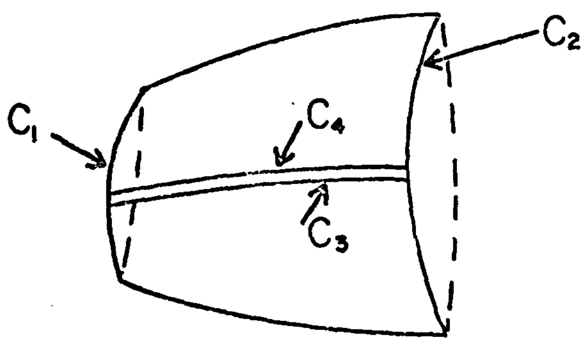
9. R. A. Graves, "Application of a Numerical Orthogonal Coordinate Generator to Axisymmetric Blunt Bodies," NASA Tech. Memo. 80131 (1980).

10. L. P. Eisenhart, "A Treatise on the Differential Geometry of Curves and Surfaces," Ginn and Company, N.Y. (1937).

11. Z. U. A. Warsi and J. F. Thompson, "Machine Solutions of Partial Differential Equations in the Numerically Generated Coordinate Systems," Engineering and Experimental Research Station, Mississippi State University, Rep. No. MSSU-EIRS-ASE-77-1 (1976).



(a)



(b)

Figure 1: (a) Topology of the given surfaces. Inner $\eta = \eta_B$, outer $\eta = \eta_\infty$, current variables ξ, ζ. (b) Surface to be generated for each ζ = const., current variables ξ, η.

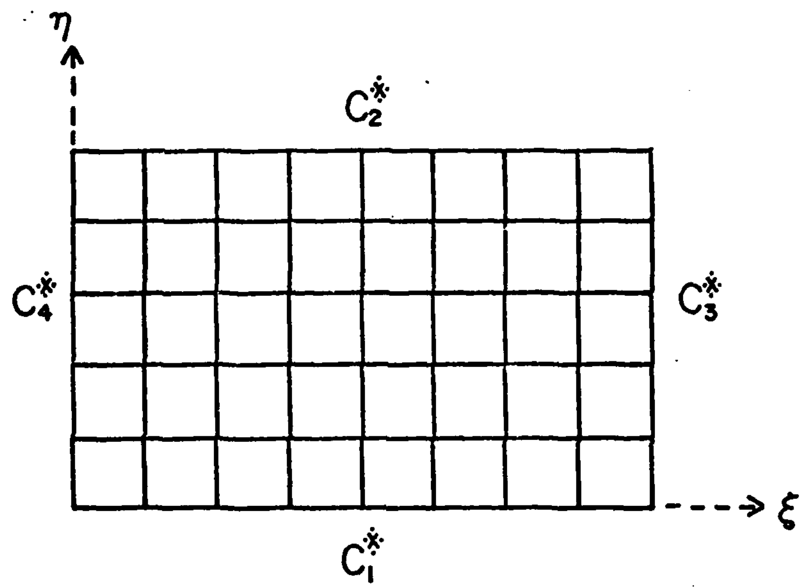


Figure 2: Figure 1(b) opened in a rectangular plane by imagining a cut.

GRID GENERATION FOR AIRCRAFT CONFIGURATIONS*

C. W. Mastin and J. F. Thompson
Mississippi State University
Mississippi State, MS 39762

*Research Sponsored by Air Force Office of Scientific Research under Grant No. AFOSR 80-0185

Introduction

The problem of generating a computational grid about an aircraft configuration has recently been addressed by Ericksson [4] and Lee [7]. In this report the basic steps in a grid generation scheme will be described. No particular aircraft is considered, but the procedure is intended to be versatile and able to handle many types of configurations with only slight changes in the computational space. The primary components of an aircraft are generally the wings and fuselage. Therefore, these components determine the basic structure of the computational grid.

The first, and one of the most difficult, task encountered is the description of the surface coordinates. In the design of various geometric objects, often paramatric surface patches are joined along specific curves on the surface of the object. These patches may be familiar quadratic surfaces, which will be considered here, or surfaces defined by interpolating polynomials or splines. Of course, the shape of these surface patches will depend on the type of grid which is to surround the body. The surface parametrization induces a natural grid system on the body which may not be best for computational purposes. Thus a method for reparameterizing the surface patch will be described. This leads to considerable control over the distribution of surface grid points.

For the general configuration being considered, a single rectangular computational region will be used. A better distribution of grid points can often be obtained by adding or subtracting rectangular blocks from the computational region as noted by Lee et al. [7]. However, the basic topology of the grid structure remains the same. Slits in the computational region may be used to represent the horizontal and vertical stabilizers and any other fin-like components. The nacelles may be incorporated by removing rectangular blocks from the computational region.

The physical region about the aircraft, from a topological viewpoint, is obtained by folding the computational region about the body. This folding results in mesh points with irregular neighborhood structures. Although the transformation from the computational to the physical region is singular at such points, no difficulty is encountered in deriving accurate difference equations for approximating a partial differential equation.

The grid generation schemes discussed here are algebraic with emphasis on the control of the grid lines. The values of the grid point coordinates can be used as initial data in an elliptic system which would smooth out any discontinuities in grid line tangents or ripples caused by the body or interpolating functions. No particular interpolation procedure will be stressed. Rather, a general format will be followed which permits inclusion of polynomial interpolation, splines, and the multi-surface method developed by Eiseman [2,3].

Coordinate System Defined by the Grid

The arrangement of the grid lines can be visualized by considering a curvilinear coordinate system about the aircraft body. For fluid flow calculations it is desirable to have the majority of the grid points located near the body and in the wake region. A curvilinear coordinate system may be generated by folding a rectangular region about the body. For example, a first fold can be made around the front of the body with edges coinciding with the wing tips and proceeding downstream. Folds at each wing tip then result in a region enclosing the aircraft as illustrated in Figure 1. The lines from the aircraft to the outer boundary are lines along which the coordinate system is singular. A more detailed discussion of this type of coordinate system has been given by Ericksson [4]. This construction results in an ellipsoidal-cylindrical coordinate system surrounding the body. The elliptical portion gives good resolution of the area around the body while the cylindrical portion is used to resolve the wake region.

Surface Grid Points

The coordinates of the grid points on the surface of the aircraft and on the outer boundary must be compatible with the curvilinear coordinate system surrounding the aircraft. This will be illustrated by considering a simple wing-fuselage configuration. All components are defined using linear and quadratic equations and the shape of the body can be changed by redefining only a few basic parameters. Two particular variations are given in Figures 2 and 3. The outer boundary is depicted in Figure 4.

Even though the surface components are relatively simple, one must be able to select the grid points according to the requirements of the problem being solved. For example, one may wish to have a uniform distribution of mesh points or to have a concentration of mesh points in regions of particular interest. The general problem of determining grid points on an arbitrary surface given by parametric equations will now be addressed.

Suppose a surface is given by the parametric equation

$$V = V(s,t)$$

where $V = (x,y,z)$ and $0 \leq s,t \leq 1$. The choice of a finite number of s and t values will determine the grid points on the surface. This choice will depend on the desired distribution of grid points. Although interpolation on a table of values for s and t could be used, the following procedure is very simple and effective. We first consider the choice of s value for a fixed t. A change of variables to a new parameter ξ, with $s = s(\xi)$, $s'(\xi) > 0$, will be assumed so that equally spaced ξ values can be used in the computational region. The distribution of grid points along the t = constant curve can be specified by the arc length derivative

$$\ell(\xi) = |V_\xi(s(\xi),t)| = |V_s(s,t)|s'(\xi).$$

Thus $s(\xi)$ must satisfy the differential equation

$$s(\xi) = \ell(\xi)|V_s(s,t)|^{-1} \tag{1}$$

which can be solved by any standard numerical method once an initial condition is specified. In the numerical solution, the value $\ell(\xi)$ is approximated by the desired spacing between mesh points. The following two-dimensional example will clarify the general procedure. Let a quadrant of an ellipse be given by

$$x = a \cos s, \quad y = b \sin s, \quad 0 \geq s \geq \frac{\Pi}{2}$$

suppose equally spaced points are desired. The length of the curve is approximately

$$L = \frac{\Pi}{2}\sqrt{\frac{a^2 + b^2}{2}} \quad .$$

Thus the length of the arc between any two grid points is approximately $K = L/(N-1)$ where N is the number of grid points. Equation (1) then becomes

$$s'(\xi) = K[a^2 \sin^2 (s(\xi)) + b^2 \cos^2(s(\xi))]^{-\frac{1}{2}}, \ \xi_0 \leq \xi \leq \xi_1,$$

with initial condition $s(\xi_0) = 0$. Due to the error in the numerical solution and the approximation of arc length, some scaling was necessary in order that $s(\xi_1) = \frac{\pi}{2}$. Even with this rescaling, the uniform distribution of grid points along the central vertical coordinate line in Figure 4 attests to the effectiveness of this technique. This ellipse was used to determine one of the parameter distributions for the ellipsoidal outer boundary. Equal spaced values for the other parameter were used since this automatically concentrated grid points near the wing tips.

The grid points on the fuselage determined the position of one set of grid lines on the wings. The other set was chosen to concentrate grid points near the wing tips. The same procedure would be used for any other fin-like structure projecting from the fuselage.

Flow Field Grid

For regions with smooth boundaries, interpolation has been used extensively for grid generation in two and three-dimensional regions. Given a set of points $P_1, P_2, \ldots, P_n$, and a set of parameter values $s_1, s_2, \ldots, s_n$, a curve passing through the points P_i, with $V(s_i) = P_i$ can be represented by

$$V(s) = \sum_{i=1}^{n} \Phi_i(s)P_i$$

where Φ_i is a function satisfying $\Phi_i(s_j) = \delta_{ij}$.

While Lagrange polynomials could be used for the Φ_i, they frequently lead to curves with extreme oscillation and cubic splines are a more commonly used alternative. When the Φ_i are cubic splines, their derivatives at s_1 and s_n may be given arbitrarily. Thus the direction of the tangent vector to the curve at P_1 and P_n can be specified. Cubic splines have continuous second order derivatives which produces curves sufficiently smooth for most grid generation problems. In the following applications the points P_i, $i = 2, \ldots, n-1$, often do not lie on the body and are used only to control the general direction of the curve. In this case the condition on the Φ_i above can be replaced by

$$\Phi_i(s_j) = \delta_{ij} \text{ for } j = 1 \text{ or } n \text{ and } \sum_{i=1}^{n} \Phi_i(s) = 1,\ \Phi_i(s) \geq 0$$

Examples of functions satisfying these conditions are the B-spline basis functions developed by de Boor [1] and used by Gordon and Riesenfeld [6] and the basis functions used by Eiseman [3] in his multisurface method. The latter development is more general and could include basis functions other than piecewise polynomials. It was observed that the piecewise quadratic curves of Eiseman are the same as would be generated using B-splines.

Returning to the general configuration in Figure 1, all that is presently given is the grid points on the aircraft surface and on the ellipsoidal outer boundary. Next a surface of grid points will be constructed in the wake region behind the aircraft. This will employ interpolating functions. A set of intersecting curves is constructed as in Figure 5 to connect the aircraft with the downstream boundary. These are interpolation curves which are parameterized using surface parameters s and t so that $\Gamma_i = V(s,t_i)$ and $\Lambda_j = V(s_j,t)$. Once the basis functions are selected the two-dimensional blended interpolant of Gordon and Hall [5] completes the generation of the grid on the wake surface. Such a surface is shown in Figure 6.

The grid points on two coordinate surfaces, say $r = 0$ and $r = 1$, have been determined. Now the space between the two surfaces, the aircraft-wake surface and the outer boundary, must be filled with grid points. This can also be accomplished by interpolation. Two intermediate surfaces are used to control the interpolating curves. The first consists of points at a fixed distance along normals to the $r = 0$ surface. The second surface contains points on the line segment between these points on the normal surface and the corresponding points on the outer surface. Thus the interior interpolation grid points are as depicted in Figure 7. Figure 8 contains plots of grids constructed about a wing-fuselage configuration. The distribution of grid points is controlled by the location of the two intermediate surfaces and the number of coordinate surfaces between each pair of intermediate and boundary surfaces. The first surface must be relatively close to the aircraft to prevent grid lines connecting the aircraft to the outer boundary from crossing.

The addition of the other components to the basic wing-fuselage configuration necessitates extra control surfaces in the interpolation process. Since some of these control surfaces extend from the aircraft to the outer boundary, a three-dimensional blended interpolant will be needed (see [5]). The control surface which would be constructed for a stabilizer is depicted in Figure 9. The construction of this surface would essentially duplicate the previous construction of the surface in the wake region. Note that computationally both surfaces of the stabilizer lie on the same control surface. The number of nodal points used in the interpolation scheme for constructing the grid above and below the stabilizers would be the same. However coordinate values on the upper surface would be used in one case while coordinate values on the lower surface would be used in the other case.

Coordinate Singularities

It was noted previously that the curvilinear coordinate system about the body has singular lines along which the Jacobian of the transformation from rectangular to curvilinear coordinates vanishes. These lines were indicated in Figure 1. Since these points have only four immediate neighboring grid points, rather than the usual six, it is probably best to delete these points from the computational process when solving fluid flow problems. If this is done the grid structure about a singular point appears as in Figure 10. Only the points on the surface cutting the singular line are shown. Now all first and second order difference expressions can be evaluated in the usual manner. This procedure is certainly simpler than using series expansions to derive separate difference equations at the singular points. Previous calculations by Mastin, Ghosh, and Thompson [8] have shown that this technique produces accurate results in potential and viscous flow problems.

References

1. C. de Boor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.

2. P. R. Eiseman, "Three-Dimensional Coordinates about Wings," Proc. AIAA Computational Fluid Dynamics Conference, Williamsburg, Va., 1979.

3. P. R. Eiseman, "A Multi-Surface Method of Coordinate Generation," J. Comput. Phys. 33 (1979) 118-150.

4. L. E. Ericksson, "Three-Dimensional Spline-Generated Coordinate Transformations for Grids Around Wing-Body Configurations," in Numerical Grid Generation Techniques, NASA CP-2166, 1980.

5. W. J. Gordon and C. A. Hall, "Construction of Curvilinear Coordinate Systems and Applications to Mesh Generation," Inter. J. Numer. Meth. Engng., 7 (1973) 461-477.

6. W. J. Gordon and R. F. Riesenfeld, "B-spline Curves and Surfaces," in Computer Aided Geometric Design (R. E. Barnhill and R. F. Riesenfeld, eds.), Academic Press, New York, 1974.

7. K. D. Lee, M. Huang, N. J. Yu, and P. E. Rubbert, "Grid Generation for General Three-Dimensional Configurations," in Numerical Grid Generation Techniques, NASA CP-2166, 1980.

8. C. W. Mastin, A. Ghosh, and J. F. Thompson, "A Fractional Step Method for the Solution of the Navier-Stokes Equations in Arbitrary Three-Dimensional Regions," in Advances in Computer Methods for Partial Differential Equations III (R. Vichnevetsky and R. S. Stepleman, eds.) IMACS, New Brunswick, N.J., 1979.

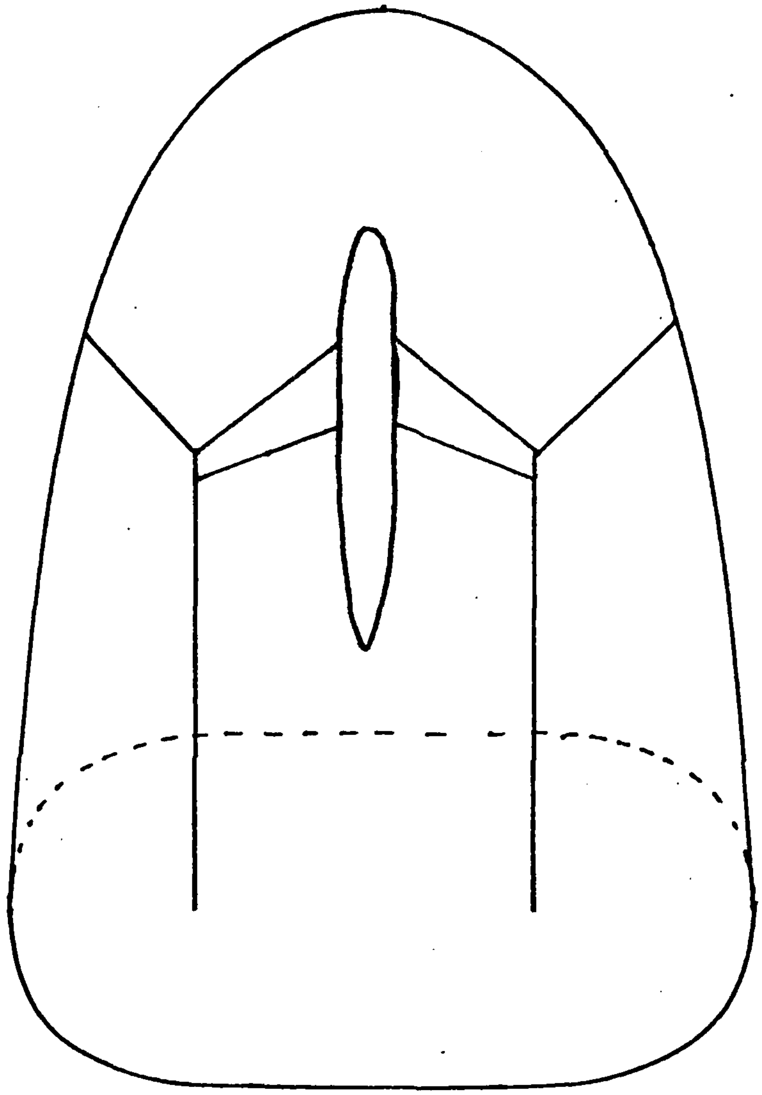


Figure 1. Aircraft and flow field.

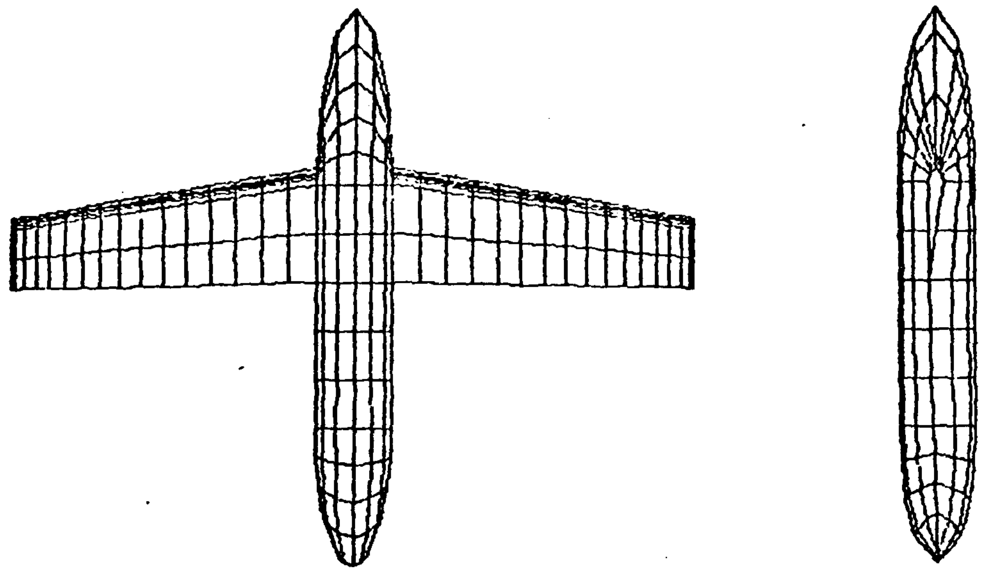


Figure 2. Wing-fuselage configuration defined by elementary surfaces.

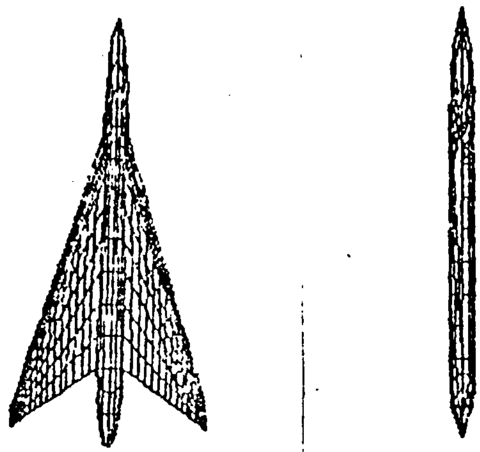


Figure 3. Wing-fuselage configuration defined by elementary surfaces.

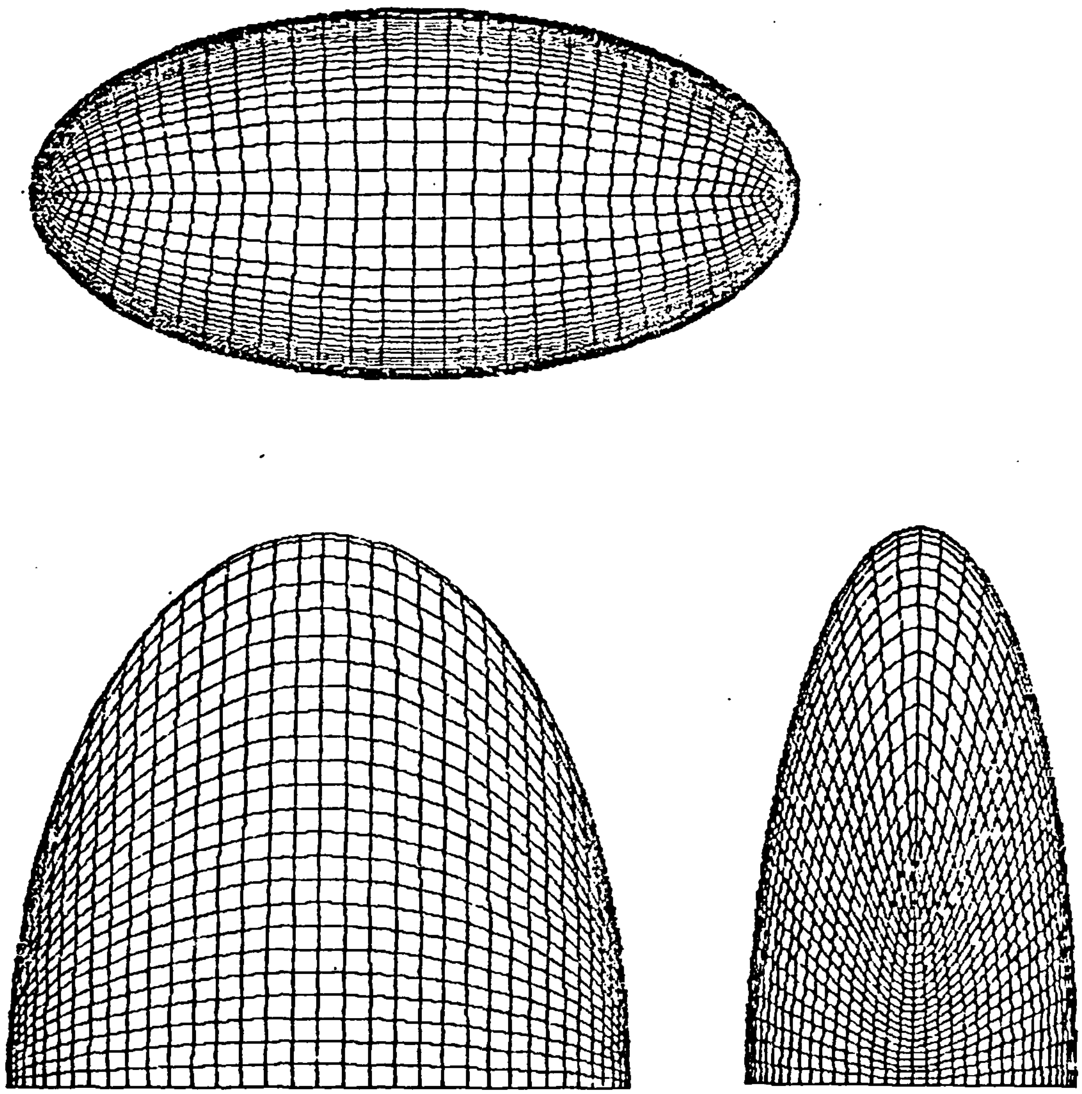


Figure 4. Outer boundary of flow field.

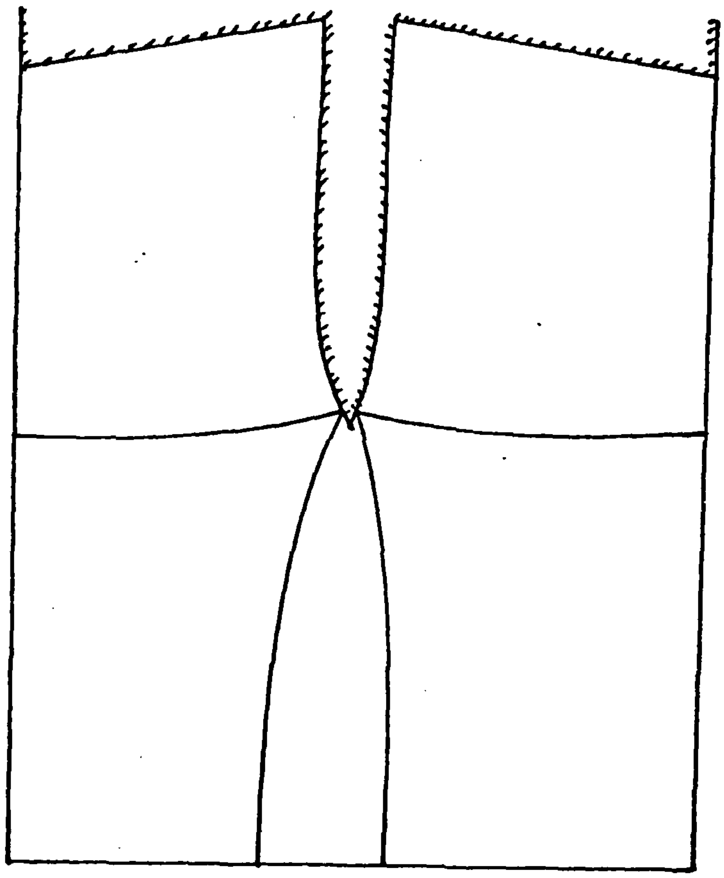


Figure 5. Interpolation curves for wake surface.

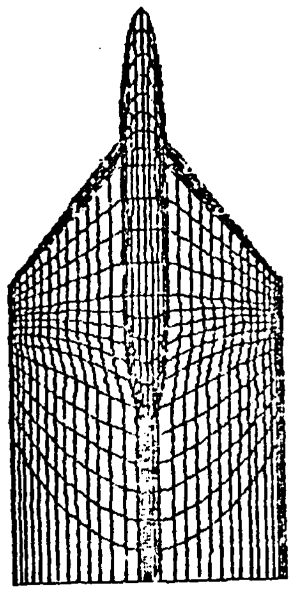


Figure 6. Grid on aircraft and surface extending to the end of the wake region.

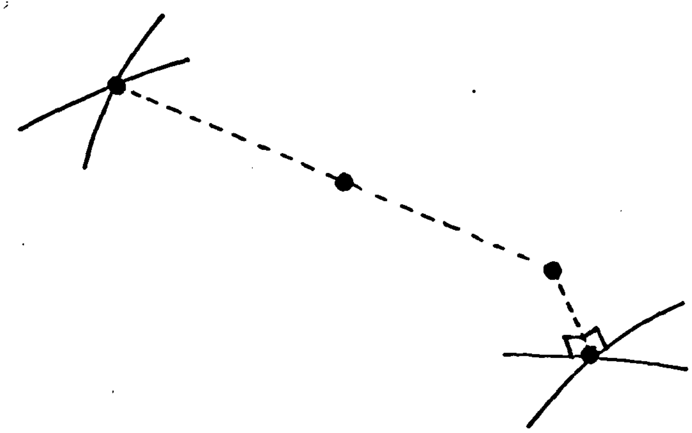


Figure 7. Interpolation points between aircraft surface and outer boundary.

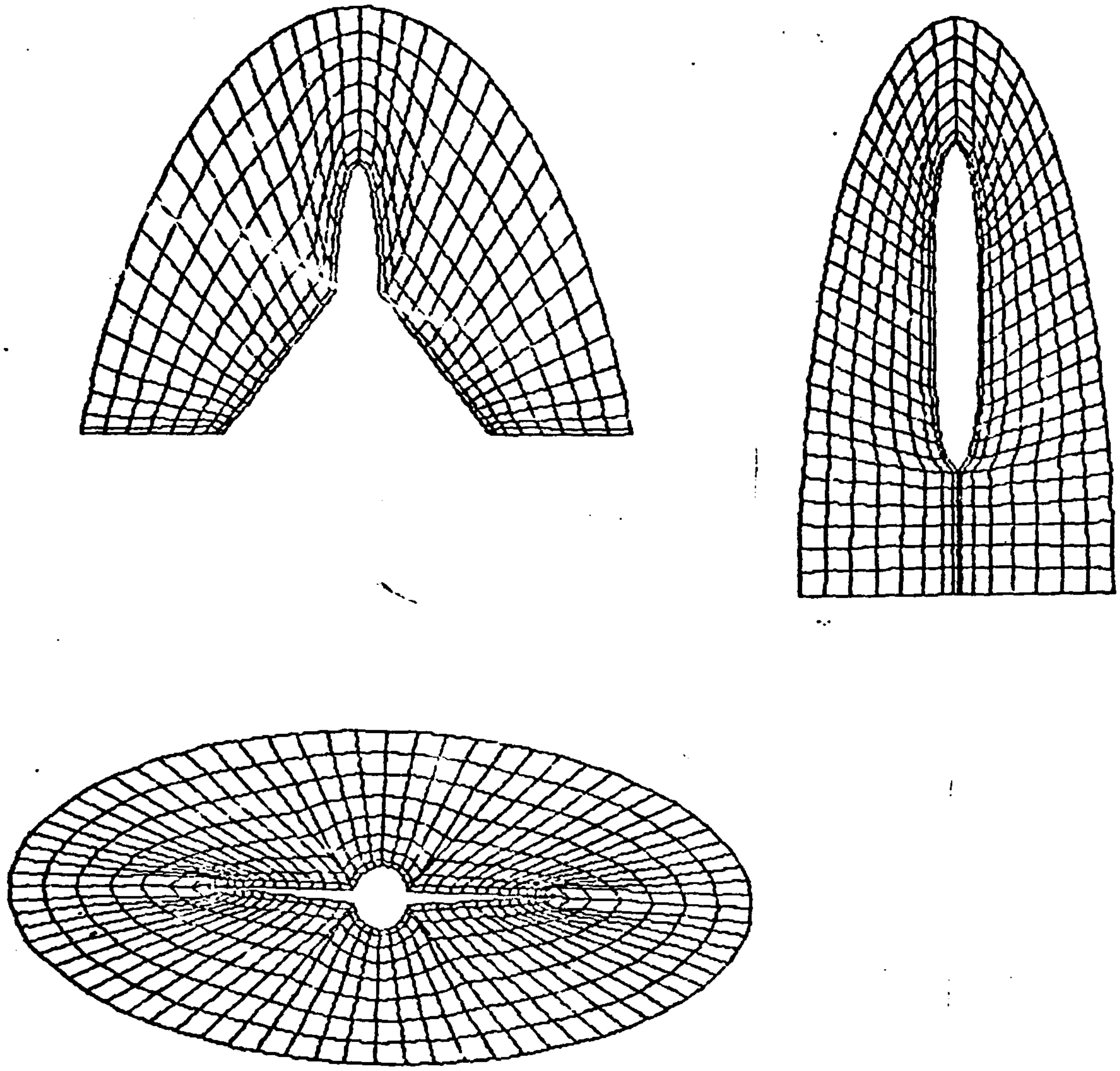


Figure 8. Crossections of grid about aircraft.

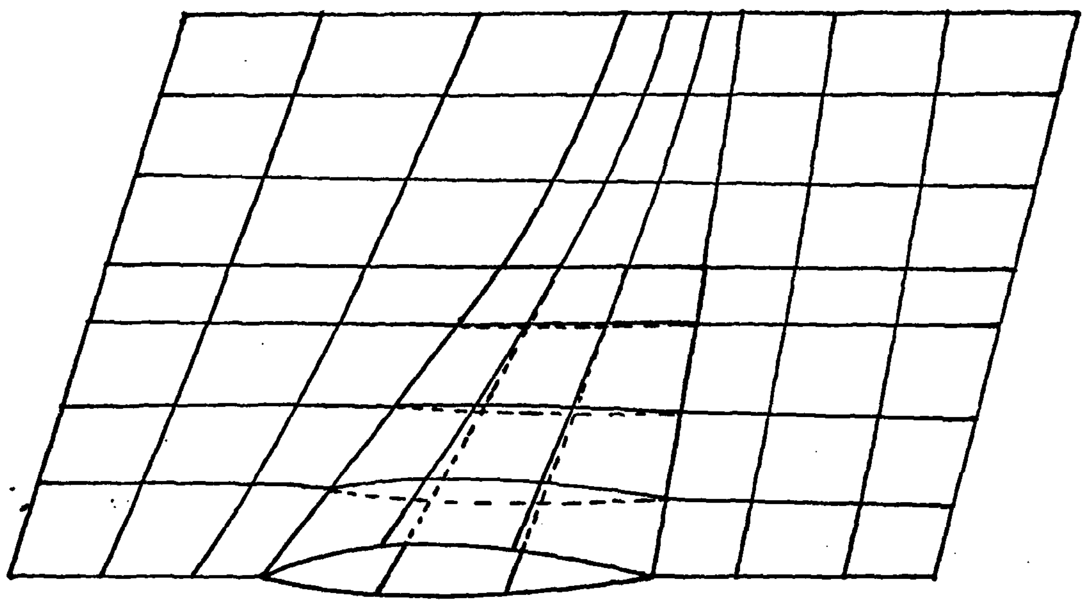


Figure 9. Grid on interpolation surface for stabilizer.

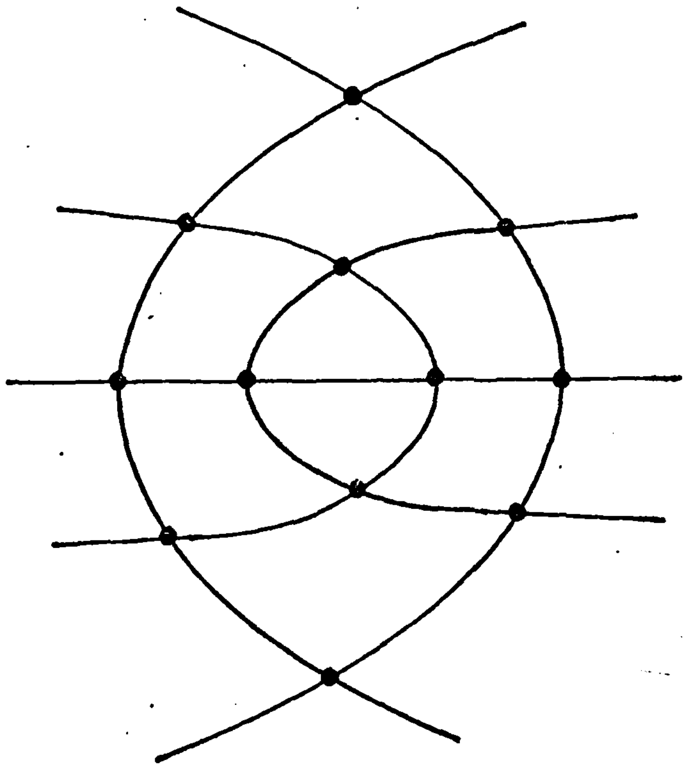


Figure 10. Grid structure on surface near intersection with singular line.